Class 15, given on Feb 5, 2010, for Math 13, Winter 2010

1. Some more examples of line integrals

Recall that the general procedure for evaluating a line integral is to first determine a parameterization $\langle x(t), y(t)\rangle, a \le t \le b$, for the curve C you are integrating over, and then plug in that information into the equation

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

The result is a definite integral in the variable t, which you evaluate using whatever methods you know. Let's look at a few more examples involving line integrals.

Examples.

- Let C be the line segment connecting $(1,1)$ and $(3,5)$, and let $f(x,y) = xy$. Evaluate $\int_C f(x,y)\,ds$.

  We start by finding a parameterization of C. There are a variety of ways to do this. For example, the line segment between two points P_0, P_1 always has a parameterization

  $$\ell(t) = P_0(1-t) + P_1 t, 0 \le t \le 1.$$

  where we treat P_0, P_1 as vectors in this equation. In this problem, this gives a parameterization of

  $$\ell(t) = \langle 1,1\rangle(1-t) + \langle 3,5\rangle t = \langle 1+2t, 1+4t\rangle, 0 \le t \le 1.$$

  Therefore, the line integral of $f(x,y)$ over C is

  $$\int_C f(x,y)\,ds = \int_0^1 (1+2t)(1+4t)\sqrt{2^2+4^2}\,dt = \int_0^1 \sqrt{20}(8t^2+6t+1)\,dt = \sqrt{20}\left(\frac{8}{3}+3+1\right) = \frac{20\sqrt{20}}{3}.$$

  Of course, you can choose a different parameterization for C. For example, if you determine that C lies on the line $y-1 = 2(x-1)$, or $y = 2x-1$, then you might select the parameterization $x(t) = t, y(t) = 2t-1, 1 \le t \le 3$. Then the corresponding calculation of the line integral of f over C is

  $$\int_C f(x,y)\,ds = \int_1^3 t(2t-1)\sqrt{1^2+2^2}\,dt = \int_1^3 \sqrt{5}(2t^2-t)\,dt$$
  $$= \sqrt{5}\left(\frac{2t^3}{3} - \frac{t^2}{2}\Big|_{t=1}^{t=3}\right) = \sqrt{5}((18-9/2)-(2/3-1/2)) = \frac{40\sqrt{5}}{3},$$

  which is exactly the same answer we computed earlier. This illustrates the principle that it does not matter which parameterization of C you choose, so long as you select a parameterization which traverses each point of C exactly once.

- We can also calculate line integrals over curves C which lie in $\mathbb{R}^3$ (or, for that matter, in any $\mathbb{R}^n$). The formula

  $$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

  still holds true for any parameterization $\mathbf{r}(t), a \le t \le b$ of C. In this situation, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ will have three components, so $|\mathbf{r}'(t)| = \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}$.

For example, let C be given by the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle, 0 \le t \le 2\pi$, so that C is one coil of a helix. Suppose that a thin wire has the shape of C, and has density $\rho(x, y, z) = z$. What is the mass of this wire?

The mass is given by the line integral

$$\int_C \rho\, ds = \int_0^{2\pi} z(t)\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Since $x'(t)^2 + y'(t)^2 + z'(t)^2 = (-\sin t)^2 + (\cos t)^2 + 1^2 = 2$, and $z(t) = t$, this integral is

$$\int_0^{2\pi} t\sqrt{2}\, dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2.$$

2. Vector fields

We've briefly seen how to calculate line integrals of real-valued functions over various curves C, which can lie either in $\mathbb{R}^2$ or $\mathbb{R}^3$. These integrals were motivated by various real-life applications: for example, a line integral of f over C can be interpreted as the (signed) surface area of a surface lying over C with height f, or as the mass (or total electric charge) of a wire whose density is given by f.

However, a full treatment of the theory of line integrals requires that we learn how to integrate not just real-valued functions, but also vector-valued functions. The most obvious application of this is to the calculation of the work done on a particle by a force. Recall that in high school physics one learns $W = Fd$, so long as the force and the direction an object moves are parallel to each other. After learning about vectors, one learns that $W = \mathbf{F} \cdot \mathbf{d}$, so that we can calculate work even if the force and direction are not parallel to each other. What happens if the direction of motion as well as the force are constantly changing? For example, perhaps a particle moves along a curve C, and the force acting on it varies on C. How much work does the force do to the particle then? This is the type of question line integrals were designed to answer.

Before we actually do any integrals we will briefly discuss vector fields, which are the objects we will integrate. A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, or more generally, a function $\mathbf{F} : D \to \mathbb{R}^n$, where D is a subset of $\mathbb{R}^n$. In this class we will deal almost exclusively with the cases $n = 2, 3$. We think of vector fields as functions which assign a vector to each point of $\mathbb{R}^n$. For example, if $n = 3$, we think of a vector field as being a function which assigns to each point (x, y, z) a vector $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. From now on we will reserve the (capital) letters P, Q, R for the components of a vector field. If we are looking at vector fields on $\mathbb{R}^2$, we just use $P(x, y), Q(x, y)$.

We graphically represent vector fields by plotting the vectors $\mathbf{F}(x, y)$ at a selection of points, with their tails starting at the point (x, y) in question. Such sketches do not contain all the information about $\mathbf{F}$, since we are only plotting some finite number of vectors (contrast this to when you sketch the graph of a function $y = f(x)$, where you plot every value of $f(x)$ on an interval), but often can provide a good intuitive idea of the behavior of the vector field. When drawing vector fields it is important that you get the relative length of various vectors correct, as well as the direction of those vectors.

Examples.

- Plot the vector field $\mathbf{F}(x, y) = \langle 1, 1\rangle$. This is a vector field which assigns the vector $\langle 1, 1\rangle$ to every point of $\mathbb{R}^2$. So a sketch of this vector field looks like a sketch of lots of parallel vectors, each of the same length.

- Plot the vector field $\mathbf{F}(x,y) = \langle x, y\rangle$. This is a vector field whose vectors point *radially outward*; that is, they all point away from the origin. Furthermore, the length of a vector is equal to its distance from the origin, so these vectors get larger and larger the further from the origin we get.
- Plot the vector field $\mathbf{F}(x,y) = \langle y, -x\rangle$. This is a vector field whose vectors are perpendicular to the line segment connecting (x,y) to the origin. To see this, take the dot product of $\langle x, y\rangle$ with $\langle y, -x$. This dot product equals 0, which means these two vectors are orthogonal. If we draw in circles $x^2 + y^2 = r^2$, for various r, we see that these vectors are tangent to the circles, and that these vectors point in the counterclockwise direction along these circles. The magnitude of these vectors is equal to the distance of (x,y) from the origin, and so increases as we get further out from the origin.
- Plot the vector field $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$. Again, these vectors point radially outward. However, their lengths are all equal to 1, so while the direction of these vectors change with (x,y), their magnitude does not. Notice that this vector field is not defined at $(0,0)$, and cannot be defined at $(0,0)$ in any way so as to make the vector field 'continuous'.
- The most common application of vector fields in physics and engineering is to the description of gravitational or electric fields. For example, suppose we place a positive charge of 1 unit at the origin. Suppose we place some other charge, of charge q, at a point (x,y). Then Coulomb's Law says that the force the second charge feels due to the first is given by

  $$\frac{q}{d^2}$$

  where d is the distance between the two charges, and the direction of the force is pointing away from the origin. If $q > 0$, this corresponds to the fact that two like charges repel, and if $q < 0$, this corresponds to the fact that two opposite charges attract. (This does not look exactly like the expression you might see in physics classes or texts, where some constants can appear. We are selecting our units in such a way to ensure that all relevant constants equal 1.) We can re-express this using vectors by saying that the force is equal to the vector

  $$\mathbf{F} = \frac{q}{d^3}\mathbf{d},$$

  where $\mathbf{d}$ is the vector $\langle x, y\rangle$; ie, the vector which starts at the origin and ends at the second charge. We define the electric field that the charge at the origin produces to be the field $\mathbf{E}$ defined by $\mathbf{E}q = \mathbf{F}$; that is, given an electric field, we can calculate the force the particle responsible for the field exerts on another particle by simply multiplying the charge of that particle by the field. The electric field generated by our original particle is directed radially outward, with vectors of magnitude $1/d^2$. This explains why vector fields which point radially outward or inward are so important – because they appear in nature. Again, notice that there is no way to define this field at the origin in such a way so as to make the vector field continuous.
- Another physical interpretation of a vector field is as the rate of flow of a fluid. For example, a vector field $\mathbf{F}(x,y) = \langle e^x, 0\rangle$ could represent a fluid which flows exponentially faster as x increases. The vector $\mathbf{F}(x,y)$ represents the amount of fluid which is passing through (x,y) per unit time.

3. Gradients and vector fields

Suppose f is a scalar function on $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, suppose $f(x, y)$ is defined on $\mathbb{R}^2$, and is differentiable everywhere. Then the gradient of $f(x, y)$ is defined by $\nabla f(x, y) = \langle f_x(x, y), fy(x, y)\rangle$. Notice that this is also a vector field!

In general, we call any vector field $\mathbf{F}$ which is also equal to the gradient of some function a *gradient vector field* or *conservative vector field*. If $\mathbf{F}$ is conservative, with $\mathbf{F} = \nabla f$, we sometimes call f a *potential function* for $\mathbf{F}$.

Examples.

- The electric field generated by a particle is conservative. One can check that the function $f(x, y) = -(x^2+y^2)^{-1/2}$ is a potential function for $\mathbf{E}$ by direct calculation. The physical interpretation of $f(x, y)$ is that it represents the potential energy of the electric field at a point (x, y). For example, a particle very far from the origin should have high potential energy, while a particle very close to the origin should have low potential energy, which is exactly what happens with this $f(x, y)$. We will see this interpretation of the potential function in a more precise form next week.
- On the other hand, not every vector field is conservative. For example, let $\mathbf{F}(x, y) = \langle 2y, 3x^2\rangle$. If $\mathbf{F}$ were conservative, say $\mathbf{F} = \nabla f$, then $f_x = 2y, f_y = 3x^2$. These are both continuous functions, and their partial derivatives are also continuous, so by an application of Clairaut's Theorem we would then have $f_{xy} = f_{yx}$. However, $f_{xy} = 2, f_{yx} = 6x$, which are not equal, so there is no way that this function could be conservative.

We will first learn how to integrate vector fields over curves C. We then want to investigate several questions, many of which touch on conservative vector fields. For example, do conservative vector fields have any special properties, especially with regards to line integrals? How can we determine whether or not a vector field is conservative? And what is the origin of the name 'conservative'?